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PROBABILITY OF NEGATIVE ESTIMATES OF HERITABILITY IN ONE-WAY UNBALANCED RANDOM MODEL

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Summary

The expression for the probability of getting negative estimates of heritability is derived for half sib population, one-way unbalanced random model. An approximation for the same has also been obtained using two moments approximation for the non-null distribution of between groups sum of squares by some constant times chi-square. The exact and approximate values of the probability are worked out, numerically, for certain apriori parametric values. The computed results reveal that the approximate values are close to the exact values of the probability and it increases with increase in the variability among group sizes.

Key words: Heritability, Probability of Negative Estimates, Unbalanced Random Model.

Introduction

The heritability, a genetic parameter, is a non-negative value between zero and one. The usual estimator of heritability is a function of analysis of variance estimators of variance components. The negative estimates of heritability are frequently obtained in animal breeding experiments. This is because of the occurance of negative estimates of variance components other than the error variance. This situation of negative estimates of heritability is a great concern to the geneticists.

The probability of obtaining negative estimates of heritability has been given by Gill and Jensen [1] for balanced random models in normal population. In this paper the probability of getting negative estimates of heritability has been derived for unbalanced situations in one-way random model when observations are from normal populations following the procedure used by Singh [5]. An approximation for the same has also been given by using the two-moments chi-square approximation for the non-null distribution of between groups sum of squares. The exact and approximate values for the probability of negative estimates of heritability have been worked out for some apriori parametric values and unequal group sizes.

2. *Model and Heritability Estimator*

In analysis of variance for one-way unbalanced random classification the j^{th} observation of the i^{th} group (y_{ij}) is expressed by an equation (model).

$$y_{ij} = m + a_i + e_{ij} \tag{2.1}$$

$$j = 1, 2, \ldots, n_i \,;\; i = 1, 2, \ldots, k \,;\; N = \sum_{i=1}^{k} n_i$$

where m is the general mean; a_i, the random effects due to k groups, are iid normal with zero mean and variance σ_a^2 ; e_{ij}, the error variables independent of a_i are iid normal with zero mean and variance σ_e^2, k is the number of groups and N is the total number of observations. The variances σ_a^2 and σ_e^2, are known as components of group and error variance, respectively.

The between groups sum of squares is defined as

$$SSB = \sum_{i=1}^{k} n_i \,(\bar{y}_i - \bar{y})^2 \tag{2.2}$$

and the within groups sum of squares is defined as

$$SSE = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2 \tag{2.3}$$

where $\bar{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}$ and $\bar{y} = \frac{1}{N} \sum_{i=1}^{k} n_i \bar{y}_i$ are means.

The usual estimator of heritability (half sib) is defined as

$$\hat{h}^2 = 4\,\hat{\sigma}_a^2/(\hat{\sigma}_a^2 + \hat{\sigma}_e^2) \tag{2.4}$$

where $\hat{\sigma}_a^2$ and $\hat{\sigma}_e^2$ are usual ANOVA estimators of variance components σ_a^2 and σ_e^2, respectively and are given by

$$\hat{\sigma}_a^2 = (\,MSB - MSE\,)/k' \text{ and } \hat{\sigma}_e^2 = MSE \tag{2.5}$$

with $MSB = SSB/p$ and $MSE = SSE/q$ as the corresponding mean squares with $p = (k\text{-}1)$ and $q = (N - k)$ degrees of freedom, respectively, and $k' = (N - \sum_{i=1}^{k} n_i^2/N)/p$.

Then, using (2.5), the estimator $(\hat{h}^2)$ of heritability is expressed as

$$\hat{h}^2 = 4\,(MSB - MSE) \,/\, [\, MSB - (k' - 1)\, MSE\,] \tag{2.6}$$

3. *Probability of Negative Estimate of* h^2

The probability of negative estimates of h^2 can be expressed as

$$p\,(\,\hat{h}^2 < 0) = P\left[\frac{MSB}{MSE} < 1\right]$$

$$= P\,(\,F < 1\,), \tag{3.1}$$

where F is the usual variance ratio in one-way classification. This approach is also used by Singh [5] for deriving the probability of negative estimates of σ_a^2, the group variance component. For evaluating (3.1) one needs the non-null distribution of F which is given as under

Denote $U_\bullet = \sigma_e^2\, I_p + \sigma_a^2\, M_\bullet$, where I_p is an identity matrix of order p, $M_\bullet$ is a non-singular matrix of order p such that its eigen values are the non-zero eigen values of the matrix $M = (\,m_{ii'}\,)$ with

$$m_{ii'} = n_i\left(1 - \frac{n_i}{N}\right), \quad i' = i$$

$$= -\,n_i\, n'_i / N\,, \quad i' \neq i\,,$$

$$i, i' = 1, 2, \ldots, k\,.$$

Notice that there are many choices of $M_\bullet$ and one choice is $M_\bullet = (\,m_{jj'}\,)$, where

$$m^\bullet_{jj'} = n_j + \frac{n_j}{N}\,(n_i - n_j)\,, \quad j' = j$$

$$= n'_j\,(n_i - n_j)\,, \qquad j' \neq j$$

$j, j' = 2, 3, \ldots, k$

With this, the non-null distribution of F is given by Singh [6] as

$$P[F \le x] = \sum_{j=0}^{\infty} e_j \; p_r \left[\left(\chi^2_{p+2j}/\chi^2_q\right) \le (p\sigma^2_e/q\beta)\, x\right], \tag{3.2}$$

where β is an arbitrary constant such that (3.2) is convergent and is taken as $\beta = \sigma^2_e (1 + n_o \lambda)$ for practical applications, $n_o = \min n_i$ and

$$\lambda = (\sigma^2_a/\sigma^2_e) = h^2/(4 - h^2)$$

The coefficients (e_j) are defined as

$$e^2_o = \beta^p \mid U_* \mid^{-1}, \quad e_r = (2r)^{-1} \sum_{j=0}^{r-1} e_j H_{r-j}, \quad r \ge 1$$

with $H_s = t_r (I_p - \beta U_*^{-1})^s$ and

t_r denotes the trace of a matrix.

The relation between the distribution of ratio of two independent central chi-square variables or the distribution of F and the beta distribution (Rao [3], p 167) is used to write

$$P_r[(\chi^2_{p+2j}/\chi^2_q) \le (p\sigma^2_e/q\beta)\, x]$$
$$= 1 - I_w[q/2, (p/2) + j], \tag{3.3}$$

where

$$I_w(a, b) = \frac{1}{B(a,b)} \int_0^w t^{a-1} (1 - t)^{b-1} \, dt$$

and

$$w = q\beta/[q\beta + (p\,\sigma^2_e)\, x]$$

In this way, we write (3.2) as

$$p[F \le x] = 1 - \sum_{j=0}^{\infty} e_j \; I_w[q/2, (p/2) + j] \tag{3.4}$$

and, one can easily evaluate P [$\hat{h}^2 < 0$] with the help of expressions (3.1) and (3.4) after substituting x = 1 and using tables on the beta distribution (see Pearson and Hartley, [2], pp 32-33).

4. *An Approximation*

The expression (3.2) for obtaining the probability of getting negative estimates of heritability is tedious for practical applications. We, therefore, obtain an approximation for the same. For this, we first obtain a two moments approximation for the non-null distribution of between groups sum of squares SSB as $u\chi_f^2$, usually adopted in analysing random effects models (see Searle, [4], where u and f are constants, obtained from the equations derived by equating the first two moments of SSB and $u\chi_f^2$. The equations are

$$u f = E\,(SSB)$$

and $$2\,u_f^2 = \text{Var}\,(SSB) \tag{4.1}$$

where E(SSB) and Var (SSB) (see Tan [7] equation 3.1) can be expressed in terms of heritability (h^2) as

$$E\,(SSB) = p\,(\sigma_e^2 + k'\ \sigma_a^2) = g_1\,\sigma_e^2,$$

$$\text{Var}\,(SSB) = 2\left[\ \sum_{i=1}^{k}(1-2n_i/N)\,(\sigma_e^2 + n_i\,\sigma_a^2)^2 + \Big(\sum_{i=1}^{k} n_i\,(\sigma_e^2 + \sigma_a^2)/N\Big)^2\ \right]$$

$$= g_2\,\sigma_e^4 \tag{4.2}$$

where $g_1 = p\,(1 + k'\,\lambda)$

$$g_2 = p + 2\,\Big(N - \sum_{i=1}^{k} n_i^2/N\Big)\,\lambda + \left[\ \sum_{i=1}^{k} n_i^2\,(1 - 2n_i/N) + \left(\sum_{i=1}^{k} n_i^2/N\right)^2\right]\lambda^2$$

are functions of h^2 only in addition to design parameters (n_i, i =1, 2, . . . , k). The quantities u and f can thus be obtained as

$$U = \text{Var}\ (SSB)\,/2\,E\,(SSB) = (g_2/g_1)\,\sigma_e^2$$

and $$f = 2\,[\,E\,(SSB)\,]^2\,/\,\text{Var}\,(SSB) = g_1^2/g_2 \tag{4.3}$$

It is known that

$$SSE \sim \sigma_e^2 \; \chi_q^2 \tag{4.4}$$

In this way, we have that

$$F = \frac{MSB}{MSE} \;\dot{\sim}\; \frac{qu}{p\,\sigma_e^2}\; \chi_f^2/\chi_2^2 \tag{4.5}$$

Thus, using (4.5), the approximate probability of negative estimates of heritability can be obtained as

$$P(\hat{h}^2 \angle 0) = P(F \angle 1)$$

$$\doteq P\left[x_f^2/x_q^2 \le \frac{p\,\sigma_e^2}{q\,u} \right] \tag{4.6}$$

By following the steps of section 3 for deriving (3.4), the expression (4.6) can be expressed as

$$p(\hat{h}^2 \angle 0) = 1 - I_w\left(\frac{q}{2}, \frac{f}{2}\right) \tag{4.7}$$

where $W = \left(1 + \frac{\sigma_e^2 p}{q\,u}\right)^{-1}$

4. *Numerical Results*

The exact values of p $(\hat{h}^2 \angle 0)$ are worked out from (3.4) after substituting x = 1 and the corresponding approximate values from (4.7) for certain apriori values of h^2, n_i and k. To save space we present these values in Table-1 only for two situation of unbalancedness; I ($n_1 = 7$, $n_2 = 5$, $n_3 = 3$) and II ($n_1 = 3$, $n_2 = 9$, $n_3 = 3$) and some apriori values of h^2.

The computed results reveal that the approximate values are close to the exact values of p $(\hat{h}^2 \angle 0)$. Further, the value of $(\hat{h}^2 \angle 0)$ increases with increase in the variability among group sizes.

Table-1 Probability of negative estimates of h^2.

h^2	Situation I $n_1 = 7, n_2 = 5, n_3 = 3$		Situation II $n_1 = 3, n_2 = 9, n_3 = 3$	
	Exact	Approximate	Exact	Approximate
0	0.6033	0.6033	0.6033	0.6033
0.25	0.5087	0.5116	0.5205	0.5206
0.50	0.4313	0.4363	0.4493	0.4505
0.75	0.3670	0.3735	0.3880	0.3909
1.00	0.3125	0.3212	0.3344	0.3399

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